

Exploring multipartite quantum correlations with the square of quantum discord

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We explore the correlation distribution in multipartite quantum states based on the square of quantum discord (SQD). In tripartite quantum systems, the necessary and sufficient condition for the SQD to satisfy the monogamy relation is given. Particularly, our analysis shows that the SQD can be monogamous for all three-qubit pure states, and we define a genuine tripartite quantum correlation measure which can be generalized to mixed states. Furthermore, we study the quantum correlation distributions in four-qubit pure states. Although these distributions are not always monogamous, they can be used to constitute correlation indicators. As a typical application, we use these indicators in the dynamical evolution of multipartite cavity-reservoir systems, which give an effective characterization of multipartite quantum correlations in the composite system.

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I. INTRODUCTION

Characterization of quantum correlation is an important but rather challenging issue in quantum information theory. As a typical kind of quantum correlation, quantum entanglement [1] has been well understood in many aspects and widely applied to quantum communication and quantum computation [2]. However, quantum entanglement can not account for all kinds of quantum correlations [3]. It was shown that, using little or no entanglement, the deterministic quantum computation with one qubit (DQC1) [4] can perform a task exponentially faster than any classical algorithm [5, 6]. Therefore, it is believed that the quantum speedup advantage comes from quantum correlation rather than quantum entanglement in the DQC1 model. Moreover, recent studies also identify that quantum correlation is a key resource in broadcasting of quantum states [7, 8], quantum state merging [9, 10], assisted optimal state discrimination [11], remote quantum state preparation [12], and so on.

Quantum discord (QD) is a prominent bipartite quantum correlation measure [13, 14], and has been accepted as a basic tool to characterize the quantum advantage beyond entanglement. Recently, generalization of the QD to multipartite systems receives much attention. Using the relative entropy as a distance measure of correlation, Modi *et al* gave the definition of QD for multipartite systems [15]. Rulli and Sarandy proposed the global quantum discord by recasting the QD with relative entropies [16]. Moreover, with the difference of quantum versions of tripartite mutual information, two inequivalent multipartite QDs were suggested [17, 18]. However, the characterization of quantum correlation structure in multipartite systems is still very challenging. Monogamy relation of entanglement is an important property in multipartite quantum systems, which means that entanglement can not be freely shared among many parties. As quantified by the square

of concurrences [19], entanglement is monogamous in multi-qubit systems [20, 21], and this property can be used to constitute genuine multipartite entanglement measures [20, 22, 23]. Therefore, it is natural to ask whether or not the quantum correlation is monogamous, especially for the QD.

Streltsov *et al* proved that the monogamy relation does not hold in general for quantum correlation measures which are nonzero for separable states [24]. However, the quantum correlation measure can still be monogamous in special cases. For example, they showed that the geometric measure of discord [25] is monogamous in three-qubit pure states [24]. The QD is a well-defined quantum correlation measure, and its monogamy property was investigated by many authors. Prabhu *et al* found that the QD does not satisfy the monogamy even for three-qubit W states [26]. Using the Koashi-Winter formula [27], Giorgi pointed out that the condition for the QD being monogamous is equivalent to that for the entanglement of formation being monogamous [28]. While Ren and Fan showed that, under the same measurement party, the QD is still not monogamous in three-qubit pure states [29].

In this paper, we are motivated by the following two questions: (i) *whether or not* the QD is monogamous in certain form? (ii) *in what degree* the discord is monogamous and can characterize the genuine multipartite quantum correlation? To answer the two questions, we explore the monogamy property of the square of quantum discord (SQD) in multipartite quantum systems. The paper is organized as follows. In Sec. II, we derive the necessary and sufficient condition for the SQD being monogamous in tripartite quantum states. In three-qubit pure states, our results show that the SQD is monogamous, and we introduce a three-qubit correlation measure which can be further extended to mixed states. In Sec. III, we analyze the correlation distribution in four-qubit pure states and constitute a set of multipartite quantum correlation indicators. In Sec. IV, we use these indicators to characterize the dynamical evolution in the cavity-reservoir systems, which give an effective characterization for multipartite quantum correlations. Finally, in Sec. V, we discuss some problems and give a brief conclusion.

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II. MONOGAMY PROPERTY AND CORRELATION MEASURE IN TRIPARTITE QUANTUM STATES

A. Definitions and monogamous condition

In bipartite quantum systems ρ_{AB} , the total correlation can be quantified by quantum mutual information $I_{A:B} = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$ with $S(\rho) = -\text{Tr}\rho \log \rho$ being von Neumann entropy [13]. While the classical correlation is given by $J_{A:B} = \max_{\{E_j^B\}} [S(\rho_A) - \sum_j p_j S(\rho_{A|E_j^B})]$, in which $\{E_j^B\}$ is a positive operator valued measure (POVM) performed on subsystem B and $\rho_{A|E_j^B} = \text{Tr}_B(E_j^B \rho_{AB} E_j^{B\dagger})/p_j$ with $p_j = \text{Tr}_{AB}(E_j^B \rho_{AB} E_j^{B\dagger})$ [14]. The QD is used to characterize bipartite quantum correlation, which is defined as the difference between $I_{A:B}$ and $J_{A:B}$, and can be expressed as [13]

$$D_{A|B} = S(\rho_B) - S(\rho_{AB}) + \min_{\{E_j^B\}} \sum_j p_j S(\rho_{A|E_j^B}), \quad (1)$$

where the minimum runs over all the POVMs, and $D_{A|B}$ is referred to as the discord of system AB with the measurement on subsystem B . The QD can also be written in the form of quantum conditional entropy [30]

$$D_{A|B} = \tilde{S}(A|B) - S(A|B), \quad (2)$$

where the non-negative quantity $\tilde{S}(A|B) = \min_{\{E_j^B\}} \sum_j p_j S(\rho_{A|E_j^B})$ is the measurement-induced quantum conditional entropy and $S(A|B) = S(\rho_{AB}) - S(\rho_A)$ is the direct quantum generalization of conditional entropy. In the following derivation, we will use the expression in Eq. (2) for simplicity.

In tripartite pure states $|\psi\rangle_{ABC}$, the entanglement of formation [19] and measurement-induced quantum conditional entropies are related by the Koashi-Winter formula [27]

$$\tilde{S}(i|k) = \tilde{S}(j|k) = E_f(ij), \quad (3)$$

where $\tilde{S}(i|k)$ and $\tilde{S}(j|k)$ are the conditional entropies with measurements on subsystem k , and $E_f(ij) = \min_{\{p_\epsilon, \rho_{ij}^\epsilon\}} S(\rho_{ij}^\epsilon)$ is the entanglement of formation in the subsystem ρ_{ij} with the minimum taking over all the pure state decompositions $\{p_\epsilon, \rho_{ij}^\epsilon\}$ and $i, j, k \in \{A, B, C\}$. Using the Koashi-Winter formula, the quantum discord has the form

$$D_{i|k} = E_f(ij) - S(i|k), \quad (4)$$

where the discord is under the measurement on subsystem k with $i, j, k \in \{A, B, C\}$.

Monogamy relation is an important property in multipartite quantum systems. As quantified by the square of concurrences, Coffman *et al* showed that the monogamy relation $C_{A|BC}^2 \geq C_{AB}^2 + C_{AC}^2$ is satisfied in three-qubit quantum states and the residual entanglement can characterize the genuine tripartite entanglement [20]. However, for the quantum correlation, previous studies indicated that the QD is not monogamous even in three-qubit pure states [26, 28, 29].

Now we explore the monogamy property of the SQD. In a tripartite pure states $|\psi\rangle_{ABC}$, we have the relation $D_{A|BC} = S(A) = E_f(A|BC)$ in which $S(A)$ is von Neumann entropy and $E_f(A|BC)$ is the entanglement of formation under the bipartite partition $A|BC$ [13, 14]. Combining this relation with Eq. (4), we can obtain the quantum correlation distribution quantified by the SQD

$$D_{A|BC}^2 - D_{A|B}^2 - D_{A|C}^2 = M(E_f^2) + 2S(A|B)[E_f(AC) - D(A|C)], \quad (5)$$

where $M(E_f^2) = E_f^2(A|BC) - E_f^2(AB) - E_f^2(AC)$, and we use the relations $S(A|C) = -S(A|B)$ and $D(A|C) = E_f(AB) + S(A|B)$ [30]. In Eq. (5), the correlation distribution contains two terms, in which the first is an entanglement distribution relation quantified by the square of entanglement of formation and the second is a function of $S(A|B)$, $E_f(AC)$ and $D(A|C)$. Therefore, the necessary and sufficient condition for the SQD being monogamous is

$$M(E_f^2) + 2S(A|B)[E_f(AC) - D(A|C)] \geq 0. \quad (6)$$

When the condition is violated, the quantum correlation distribution is polygamous, *i.e.*, $D_{A|BC}^2 < D_{A|B}^2 + D_{A|C}^2$.

B. Monogamy property in three-qubit pure states

In the following, we will focus on the quantum correlation distribution in three-qubit pure states. We first show that the term $M(E_f^2)$ is nonnegative in the monogamous condition. In two-qubit quantum states, the entanglement of formation has an analytical expression $E_f(\rho_{ij}) = h[(1 + (1 - C_{ij}^2)^{1/2})/2]$ in which $h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$ is the binary entropy and $C_{ij} = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$ is the concurrence with the decreasing nonnegative λ_i s being the eigenvalues of matrix $\rho_{ij}(\sigma_y \otimes \sigma_y) \rho_{ij}^*(\sigma_y \otimes \sigma_y)$. As a function of the square of concurrence, the entanglement of formation obeys the following relations

$$\begin{aligned} E_f^2(C_{A|BC}^2) &\geq E_f^2(C_{AB}^2 + C_{AC}^2) \\ &\geq E_f^2(C_{AB}^2) + E_f^2(C_{AC}^2), \end{aligned} \quad (7)$$

where the monogamous relation $C_{A|BC}^2 \geq C_{AB}^2 + C_{AC}^2$ and the monotonic increase property of $E_f(C^2)$ are used in the first equation, and we use the property that E_f^2 is a convexity function of C^2 in the second equation. According to Eq. (7), we can obtain $M(E_f^2) \geq 0$.

Next, we analyze the second term in the monogamous condition via the generalized Schmidt decomposition. Under local unitary transformations, a generic three-qubit pure state can be written in the form [31]

$$|\Psi\rangle_{ABC} = \lambda_0|000\rangle + \lambda_1 e^{i\phi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \quad (8)$$

where the real numbers λ_i range in $[0, 1]$ with the condition

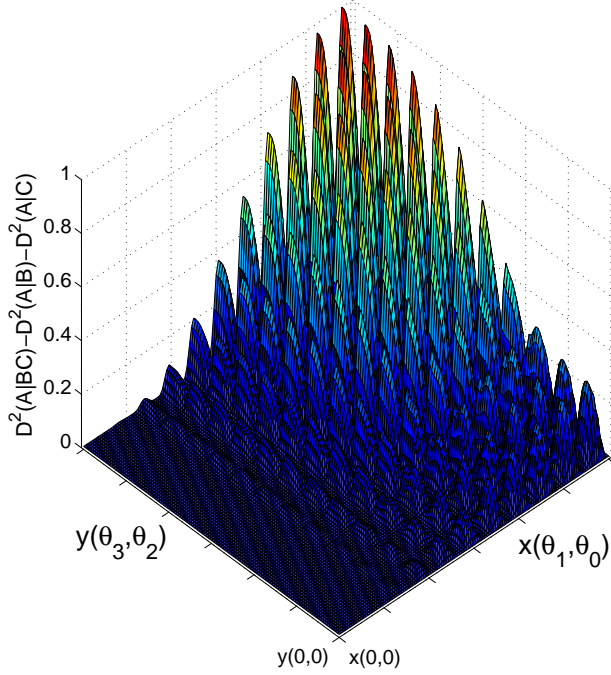


FIG. 1: (Color online) The quantum correlation distribution of SQD in the generic three-qubit pure state as a function of the parameters $x(\theta_1, \theta_0)$ and $y(\theta_3, \theta_2)$. The x -axis comes from parameters θ_0 and θ_1 , and $x(\theta_1, \theta_0)$ ranges in $[x(0, 0), \dots, x(0, \pi/2), \dots, x(\pi/40, \pi/2), \dots, x(\pi/2, \pi/2)]$. The case for $y(\theta_3, \theta_2)$ is similar. The numerical results support that the SQD is monogamous in three-qubit pure states

$\sum \lambda_i^2 = 1$, and the relative phase ϕ changes in $[0, \pi]$. Its two-qubit reduced density matrix of subsystem AB is

$$\rho_{AB} = \begin{pmatrix} \lambda_0^2 & 0 & \lambda_0 \lambda_1 e^{-i\phi} & \lambda_0 \lambda_3 \\ 0 & 0 & 0 & 0 \\ \lambda_0 \lambda_1 e^{i\phi} & 0 & \lambda_1^2 + \lambda_2^2 & g \\ \lambda_0 \lambda_3 & 0 & g^* & \lambda_3^2 + \lambda_4^2 \end{pmatrix}, \quad (9)$$

where the parameter $g = \lambda_1 \lambda_3 e^{i\phi} + \lambda_2 \lambda_4$. The density matrix ρ_{AC} is similar to ρ_{AB} , and we have the relation

$$\rho_{AC} = S_{\lambda_2 \leftrightarrow \lambda_3}[\rho_{AB}], \quad (10)$$

where $S_{\lambda_2 \leftrightarrow \lambda_3}$ is a transformation interchanging the parameters λ_2 and λ_3 . The analysis on the sign of the second term in the monogamous condition can be classified into three cases.

Case 1. The parameters $\lambda_2 = \lambda_3$. In this case, we have the density matrices $\rho_{AB} = \rho_{AC}$ and $\rho_B = \rho_C$. Therefore, the conditional entropy $S(A|B) = S(AB) - S(B) = S(C) - S(B) = 0$, and the relation $E_f(AC) - D(A|C) = E_f(AC) - E_f(AB) - S(A|B) = 0$. According to above relations, it is obvious that the second term in the monogamous equation is zero, and then the monogamous condition in Eq. (6) holds.

Case 2. The parameters $\lambda_2 \neq \lambda_3$, but the following relation is satisfied

$$|E_f(AC) - E_f(AB)| \geq |S(A|B)|. \quad (11)$$

In this case, we can derive that the second term in the monogamous condition is non-negative, and then the SQD is monogamous (see the proof in the part A of Appendix).

Case 3. The parameters $\lambda_2 \neq \lambda_3$ and the relation in Eq. (11) does not hold. In this case, the second term in the monogamous condition may be a negative value. Nevertheless, due to the first term $M(E_f^2)$ is positive, it may offset the negative value of the second term, such that the SQD is still monogamous.

The previous analysis on the monogamy property leads to a conjecture that *the SQD is monogamous in three-qubit pure states*.

Numerical proof. All the three-qubit pure states can be written in the generic form in Eq. (8) up to local unitary transformations. Therefore, it is enough to investigate the monogamy in the generic pure state. Without loss of generality, we can set $\lambda_0 = \cos\theta_0$, $\lambda_1 = \sin\theta_0 \cos\theta_1$, $\lambda_2 = \sin\theta_0 \sin\theta_1 \cos\theta_2$, $\lambda_3 = \sin\theta_0 \sin\theta_1 \sin\theta_2 \cos\theta_3$, and $\lambda_4 = \sin\theta_0 \sin\theta_1 \sin\theta_2 \sin\theta_3$ with $\theta_0, \theta_1, \theta_2, \theta_3 \in [0, \pi/2]$ and $\phi \in [0, \pi]$. We calculate numerically the quantum correlation distribution $D^2(A|BC) - D^2(A|B) - D^2(A|C)$ as a function of the five parameters $\theta_0, \theta_1, \theta_2, \theta_3$ and ϕ , where the interval of θ_i is $\pi/40$ and the case for phase ϕ is $\pi/20$. There are more than 2,100,000 quantum states in the numerical calculation and all the results is monogamous. In Fig.1, we plot the quantum correlation distribution as a function of $\theta_0, \theta_1, \theta_2$ and θ_3 (our numerical results show that the relative phase does not affect the correlation distribution and we set $\phi = 0$). The x -axis comes from parameters θ_0 and θ_1 , and the y -axis comes from parameters θ_2 and θ_3 . As shown in the figure, the distribution is zero when the parameters $\theta_i = 0, \pi/2$, which correspond to the tensor product states. For other cases, the distribution is positive. The numerical results support that the SQD is monogamous in three-qubit pure states.

C. A genuine three-qubit quantum correlation measure

A quantum correlation measure should satisfy the following necessary criteria: (i) it should be a non-negative real number; (ii) it is invariant under local unitary operations [24, 32]. For an n -partite quantum correlation measure, it should still satisfy the condition: (iii) the measure is zero if the n -partite quantum state is product in any bipartite cut [33].

Based on our previous analysis on the quantum correlation distribution of the SQD, we define a tripartite quantum correlation measure as

$$Q_3(|\psi_{ABC}\rangle) = D_{A|BC}^2 - D_{A|B}^2 - D_{A|C}^2, \quad (12)$$

which characterizes the genuine three-qubit quantum correlation in pure states. The nonnegative property of Q_3 is supported by the monogamy analysis on the SQD. The tripartite correlation Q_3 is invariant under local unitary operations due to the QD being unchanged under the transformation.

For the third requirement, we first consider the bipartite product state $|\psi_{ABC}\rangle = |\varphi_A\rangle \otimes |\varphi_{BC}\rangle$. The SQD $D_{A|BC}^2 = S^2(\rho_A) = 0$ is due to the three-qubit pure state is a prod-

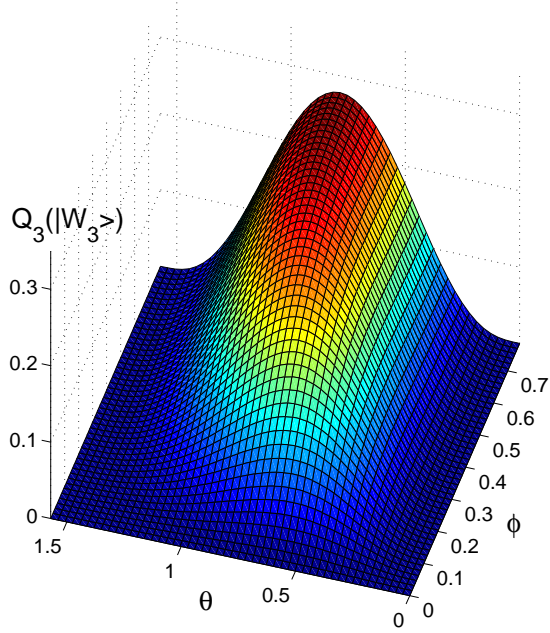


FIG. 2: (Color online) The genuine three-qubit quantum correlation Q_3 in the generalized three-qubit W state.

uct state under this partition. The SQD $D_{A|B}^2 = 0$ is because we have $(I_A \otimes E_j^B)\rho_{AB}(I_A \otimes E_j^{B\dagger}) = \rho_{AB}$ with E_j^B being the projectors composed of the eigenvectors of ρ_B . The case for $D_{A|C}^2 = 0$ is similar. So, the genuine tripartite quantum correlation $Q_3(|\psi_{ABC}\rangle) = 0$. For the product state $|\psi'_{ABC}\rangle = |\varphi_{AB}\rangle \otimes |\varphi_C\rangle$, we also have $Q_3(|\psi'_{ABC}\rangle) = 0$, since $D_{A|BC}^2 = D_{A|B}^2 = S^2(\rho_A)$ and $D_{A|C}^2 = 0$. Similarly, we can derive $Q_3(|\psi''_{ABC}\rangle) = 0$ with $|\psi''_{ABC}\rangle = |\varphi_{AC}\rangle \otimes |\varphi_B\rangle$. Therefore, Q_3 is zero when the three-qubit pure state is a product in any bipartite cut, and then the third requirement for the correlation measure is satisfied.

As an application of the correlation measure Q_3 , we consider the generalized GHZ state and the generalized W state, which are two inequivalent classes under stochastic local operations and classical communication [34]. The generalized GHZ state can be written as $|G_3\rangle = \alpha|000\rangle + \beta|111\rangle$. Its genuine tripartite quantum correlation is

$$Q_3(|G_3\rangle_{ABC}) = S^2(A) = [-\alpha^2 \log_2 \alpha^2 - \beta^2 \log_2 \beta^2]^2, \quad (13)$$

where the correlation Q_3 is equivalent to the von Neumann entropy due to $D_{A|B}^2 = D_{A|C}^2 = 0$. For the generalized W state $|W_3\rangle_{ABC} = a|001\rangle + b|010\rangle + c|100\rangle$, its three-qubit quantum correlation is

$$Q_3(|W_3\rangle) = S^2(A) - D_{A|B}^2 - D_{A|C}^2, \quad (14)$$

where $S(A) = -c^2 \log_2 c^2 - (1 - c^2) \log_2 (1 - c^2)$ and $D_{i|j} = E_f(ik) - S(i|j)$ with $i, j, k \in \{A, B, C\}$.

In Fig.2, we plot the correlation $Q_3(|W_3\rangle)$ as a function of parameters θ and ϕ , where we set the amplitudes $a =$

$\cos\theta\sin\phi$, $b = \cos\theta\cos\phi$, and $c = \sin\theta$ in the generalized W state. When the parameters $\theta = 0, \pi/2$ or $\phi = 0$, the generalized W state is a product state and its genuine three-qubit quantum correlation Q_3 is zero. In other parameter regions, $|W_3\rangle$ is a nonproduct state and the correlation is positive. As shown in the figure, the maximal value is $Q_3(|W_3\rangle) \simeq 0.2779$ which corresponds to the parameters $\theta = \phi = \pi/4$. Our result shows that the W state has the genuine tripartite quantum correlation, although its corresponding tripartite entanglement is always zero.

D. Correlation measure and indicator in mixed states

In three-qubit mixed states, the quantum correlation distribution is not always nonnegative. As an example, we analyze the quantum state $\rho_{ABC}(W) = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|$, in which the non-normalized pure state components are $|\psi_1\rangle = a|100\rangle + b|010\rangle + c|001\rangle$ and $|\psi_2\rangle = d|000\rangle$, respectively. We set the parameters $a = \cos\theta_1$, $b = \sin\theta_1\sin\theta_2\cos\theta_3$, $c = \sin\theta_1\sin\theta_2\sin\theta_3$, and $d = \sin\theta_1\cos\theta_2$. When the parameters $\theta_1 = \theta_2 = \theta_3 = 0.4\pi$, we can derive the correlation distribution $D_{A|BC}^2 - D_{A|B}^2 - D_{A|C}^2 = -0.00517$ (see the calculation in the part B of Appendix).

Although the quantum correlation distribution can be negative, we can still introduce a new measure for mixed three-qubit states via the convex roof extension of pure state correlation measure. The measure can be expressed as

$$Q_3^e(\rho_{ABC}) = \min_{\{p_i, \psi_i\}} p_i Q_3(\psi_{ABC}^i), \quad (15)$$

where the minimum runs over all pure state decompositions and $Q_3(\psi_{ABC}^i)$ is the measure in three-qubit pure states as shown in Eq. (12). According to the previous analysis on the pure state correlation measure, the mixed state measure can be nonnegative and quantify the genuine three-qubit quantum correlation.

In general, the mixed state measure Q_3^e is difficult to calculate, since it involves not only the optimal pure state decomposition but also the optimal local measurement. However, for some specific quantum states, the measure is available. For example, we consider the mixed state

$$\rho_{ABC}(p) = p|\varphi\rangle\langle\varphi| + (1-p)|\varphi^\perp\rangle\langle\varphi^\perp|, \quad (16)$$

where the components $|\varphi\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ and $|\varphi^\perp\rangle = (|000\rangle - |111\rangle)/\sqrt{2}$ and the possibility $p \geq 1/2$. In a general pure state decomposition, its pure state component has the form $|\psi(q)\rangle = \sqrt{q}|\varphi\rangle \pm \sqrt{1-q}|\varphi^\perp\rangle$ with q ranging in $(0, 1)$. For this pure state component, we can get $D_{A|B}(\psi(q)) = D_{A|C}(\psi(q)) = 0$ and $D_{A|BC}(\psi(q)) = S_A(\psi(q))$. Therefore, in the optimization of the mixed state correlation measure, we only need consider the optimal von Neumann entropy of qubit A . After some derivation, we can obtain that the mixed state correlation is $Q_3(\rho_{ABC}(p)) = h^2(1/2 + \sqrt{p(1-p)})$ with $h(x) = -x \log_2 x - (1-x) \log_2 (1-x)$.

Due to the difficulty in calculating Q_3^e , we may introduce some alternative indicators for detecting the genuine tripartite

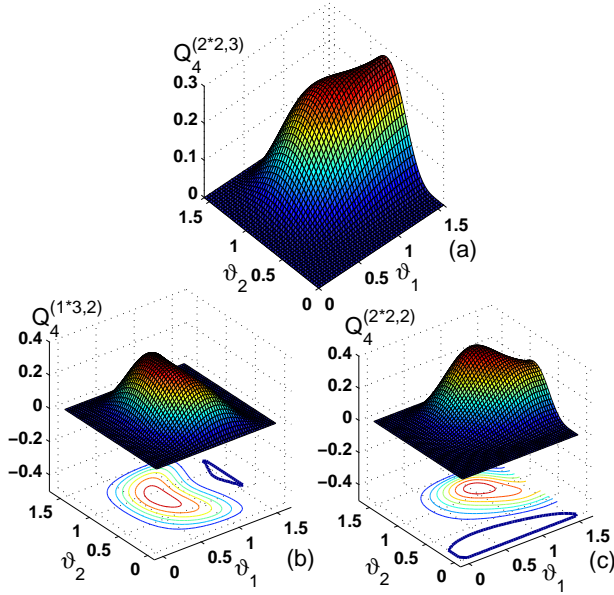


FIG. 3: (Color online) The quantum correlation indicators $Q_4(|W_4\rangle)$ as a function of the parameters ϑ_1 and ϑ_2 , in which the indicators $Q_4^{(1*3,2)}$ and $Q_4^{(2*2,2)}$ do not work in the contour region of the bold blue circle but the indicator $Q_4^{(2*2,3)}$ works in all the parameter region.

quantum correlation in the mixed state ρ_{ABC} . These indicators have the forms

$$\begin{aligned} Q_3(A|BC) &= \max[0, Q_3(A|BC)], \\ Q_3(B|AC) &= \max[0, Q_3(B|AC)], \\ Q_3(C|AB) &= \max[0, Q_3(C|AB)], \end{aligned} \quad (17)$$

where $Q_3(A|BC) = D_{A|BC}^2 - D_{A|B}^2 - D_{A|C}^2$ is the quantum correlation distribution of mixed states in the partition $A|BC$, and $Q_3(B|AC)$ and $Q_3(C|AB)$ are the mixed state distributions in partitions $B|AC$ and $C|AB$, respectively. When the indicator Q_3 is positive, it indicates that there is the genuine tripartite quantum correlation in the given mixed state. When the value is zero, the indicator does not work. It should be noted that when the indicator in partition $A|BC$ does not work, the indicators in other partitions may work and detect the multiqubit correlation. For example, in the mixed state $\rho_{ABC}(W)$, we have shown that its quantum correlation distribution in the partition $A|BC$ is -0.00517 which results in that the indicator $Q_3(A|BC)$ does not work. However, when we consider the indicator in partition $C|AB$, we have $Q_3(C|AB) = 0.08406$ which detects the multipartite correlation.

III. MULTIPARTITE QUANTUM CORRELATION INDICATORS IN FOUR-QUBIT SYSTEMS

In four-qubit pure states, the structure of quantum correlation distributions is more complicated than that in three-qubit

states. We first consider the distribution

$$Q_4^{(1*3,2)} = D_{A|BCD}^2 - D_{A|B}^2 - D_{A|C}^2 - D_{A|D}^2, \quad (18)$$

where the superscript number $1*3$ means that the correlation distribution lies in the partition between one qubit and the other three qubits, the superscript number 2 means that the correlation of subsystems is in two-qubit forms, and the $D_{A|BCD}$ is referred to as the QD with the measurement on qubits BCD .

The correlation distribution $Q_4^{(1*3,2)}$ is invariant under local unitary operations due to the quantum discords in the distribution being unchanged under this kind of operations. Moreover, for the product states $|\varphi_A\rangle \otimes |\varphi_{BCD}\rangle$ and $|\varphi_{AB}\rangle \otimes |\varphi_{CD}\rangle$, we can obtain $Q_4^{(1*3,2)} = 0$. In addition, under the qubit permutation, the correlation distribution is zero for all bipartite product states. For the generalized four-qubit GHZ state $|G_4\rangle = \alpha|0000\rangle + \beta|1111\rangle$, the correlation distribution is always nonnegative, and we have $Q_4^{(1*3,2)}(|G_4\rangle) = (-\alpha^2 \log_2 \alpha^2 - \beta^2 \log_2 \beta^2)^2$. However, the case for a generic four-qubit pure state is different, and the correlation distribution may be polygamous which results in a negative value for the $Q_4^{(1*3,2)}$. As an example, we consider the generalized four-qubit W state

$$|W_4\rangle = a|1000\rangle + b|0100\rangle + c|0010\rangle + d|0001\rangle \quad (19)$$

where the amplitudes are set to $a = \cos\vartheta_1$, $b = \sin\vartheta_1 \cos\vartheta_2$, $c = \sin\vartheta_1 \sin\vartheta_2 \cos\vartheta_3$, and $d = \sin\vartheta_1 \sin\vartheta_2 \sin\vartheta_3$, respectively. When $\vartheta_1 = 0.47\pi$ and $\vartheta_2 = \vartheta_3 = 0.28\pi$, we can derive that $Q_4^{(1,3,2)}(|W_4\rangle) = -2.1265 \times 10^{-4}$ (see the calculation in the part B of Appendix).

According to different bipartite partitions in four-qubit pure states, we have other two quantum correlation distributions which can be written as

$$\begin{aligned} Q_4^{(2*2,2)} &= D_{AB|CD}^2 - D_{A|C}^2 - D_{A|D}^2 - D_{B|C}^2 - D_{B|D}^2 \\ Q_4^{(2*2,3)} &= D_{AB|CD}^2 - D_{AB|C}^2 - D_{AB|D}^2, \end{aligned} \quad (20)$$

where the superscript $2*2$ means that the correlation distributions are considered in two-qubit partitions, and the superscript 3 means that subsystem correlations are in three-qubit states. Although the correlation distributions Q_4 s are not always nonnegative, they can still be used to constitute multi-qubit quantum correlation indicators in four-qubit pure states. In this case, we define the following four-qubit quantum correlation indicators

$$\begin{aligned} Q_4^{(1*3,2)} &= \max[0, Q_4^{(1*3,2)}] \\ Q_4^{(2*2,2)} &= \max[0, Q_4^{(2*2,2)}] \\ Q_4^{(2*2,3)} &= \max[0, Q_4^{(2*2,3)}]. \end{aligned} \quad (21)$$

When the indicator Q_4 is positive for a given quantum state, it means that there exists the genuine four-qubit quantum correlation. When the indicator is zero, it does not work and can not detect the multipartite correlation.

For the generalized four-qubit W state in Eq. (19), we use the Q_4 s to detect the genuine four-qubit quantum correlations.

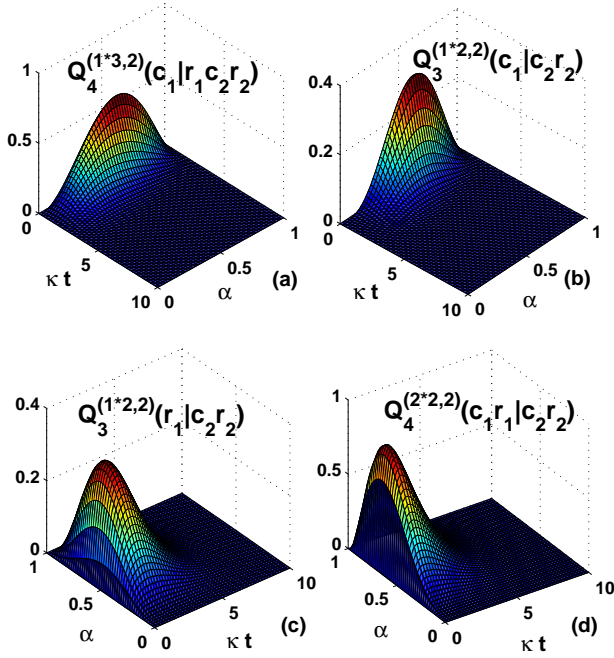


FIG. 4: (Color online) The indicators for multipartite quantum correlations in cavity-reservoir systems as a function of the time evolution κt and the initial state amplitude α .

In Fig.3, we plot the three indicators as a function of parameters ϑ_1 and ϑ_2 (we set $\vartheta_3 = \vartheta_2$). In Fig.3(a), the quantum states in the boundary correspond to the tensor product states, and we find its four-qubit correlation $Q_4^{(2*2,3)}$ is zero. The quantum states away from the boundary are non-product states, and we find that the indicator is always positive and detects the four-qubit correlation. In Fig.3(b), the indicator $Q_4^{(1*3,2)}$ is positive in most of the regions and can indicate the four-qubit correlation. In the contour region of the bold blue circle, the value of the indicator is zero, which means the indicator does not work for these quantum states. However, in the same region of Fig.3(c), the indicator $Q_4^{(2*2,2)}$ works. Similarly, when the indicator $Q_4^{(2*2,2)}$ does not work (the contour region of bold blue circle), the indicators $Q_4^{(1*3,2)}$ and $Q_4^{(2*2,3)}$ can detect the four-qubit correlation. In this sense, we argue that the four-qubit quantum correlation can not be characterized by one quantity and we need a set of quantities to characterize it from different facets. Another example is the cluster state $|C_4\rangle = (|0000\rangle + |0101\rangle + |1010\rangle - |1111\rangle)/2$ [35], in which the four-qubit correlation indicator $Q_4^{(1*3,2)} = 1$ but the indicator $Q_4^{(2*2,2)} = 2$.

All the quantum correlation indicators Q_4 are non-negative and invariant under local unitary operations. When the indicators are positive, they detect the four-qubit quantum correlation. Next, we will analyze the dynamical evolution of quantum correlation in four-partite cavity-reservoir systems, in which all the three-qubit and four-qubit indicators can characterize effectively the multipartite quantum correlations.

IV. MULTIPARTITE QUANTUM CORRELATION CHARACTERIZATION IN CAVITY-RESERVOIR SYSTEMS

Dynamical property of quantum correlations is very important in the practical quantum information processing, since quantum systems are always affected by the unwanted interaction induced by the environment. Till now, the dynamical property of two-qubit quantum correlation has been widely investigated both theoretically and experimentally (see, for example, Refs. [36–39] and references therein). However, the dynamical property of multipartite quantum correlations is still very challenging.

In this section, we consider the system composed of two entangled cavity photons being affected by the dissipation of two individual N -mode reservoirs, where the interaction of a single cavity-reservoir system is described by the Hamiltonian [40]

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\sum_{k=1}^N\omega_k\hat{b}_k^\dagger\hat{b}_k + \hbar\sum_{k=1}^Ng_k(\hat{a}\hat{b}_k^\dagger + \hat{b}_k\hat{a}^\dagger). \quad (22)$$

The initial state of the composite system is $|\Phi_0\rangle = (\alpha|00\rangle + \beta|11\rangle)_{c_1c_2}|00\rangle_{r_1r_2}$, where the dissipative reservoirs are in the vacuum state. Under the limit of large N , the output state of the cavity-reservoir system has the form [40]

$$|\Phi_t\rangle = \alpha|0000\rangle_{c_1r_1c_2r_2} + \beta|\phi_t\rangle_{c_1r_1}|\phi_t\rangle_{c_2r_2}, \quad (23)$$

where $|\phi_t\rangle = \xi(t)|10\rangle + \chi(t)|01\rangle$ with the amplitudes being $\xi(t) = \exp(-\kappa t/2)$ and $\chi(t) = [1 - \exp(-\kappa t)]^{1/2}$.

For the output state, we analyze its three- and four-partite quantum correlation indicators Q_3 and Q_4 given in Eqs. (17) and (21). In the calculation of other indicators, the multiqubit quantum discords can be obtained via the Koashi-Winter relation. However, it should be noted that the relation does not work for the calculation of some two-qubit quantum discord in the dynamical evolution, since it is involved in a solution to the entanglement of formation beyond the two-qubit case. Here, we use the method introduced by Chen *et al* for calculating the quantum discord of X states (see the calculation in the part C of Appendix) [41].

In Fig.4, we plot the multipartite quantum correlation indicators as a function of the time evolution parameter κt and the initial state amplitude α . When the time $\kappa t = 0$, the quantum state is a product state and these indicators are zero. Along with the time evolution, these indicators are positive and can detect the genuine multiqubit quantum correlations. These indicators first increase to their maximum, and then decay asymptotically. When the parameter $\kappa t \rightarrow \infty$, the output state evolves to a product state and all the multipartite quantum correlations disappear.

In the cavity-reservoir system, its multipartite entanglement evolution was investigated in Refs. [40, 42]. The genuine multipartite entanglement can be characterized by a series of entanglement indicators. Here, in our analysis, we consider

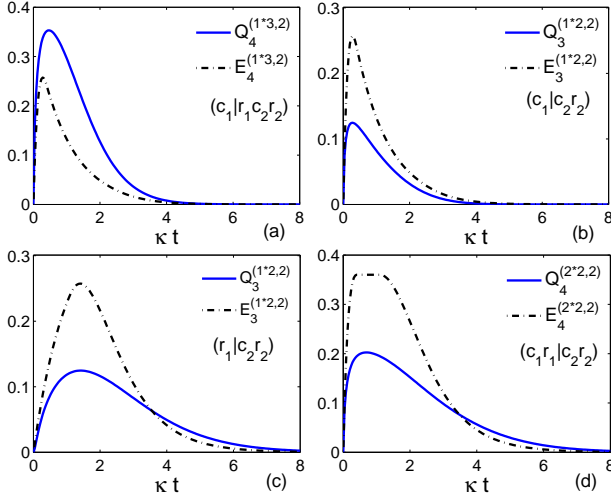


FIG. 5: (Color online) The indicators for multipartite quantum correlations as a function of the time evolution parameter κt in comparison to the multipartite entanglement indicators in the output state $|\Phi_t\rangle$ with the initial state parameter $\alpha = 1/\sqrt{10}$.

the following ones

$$\begin{aligned}
 E_4^{(1*3,2)}(|\Phi_t\rangle) &= C_{c_1|r_1c_2r_2}^2 - C_{c_1r_1}^2 - C_{c_1c_2}^2 - C_{c_1r_2}^2 \\
 E_4^{(2*2,2)}(|\Phi_t\rangle) &= C_{c_1r_1|c_2r_2}^2 - C_{c_1c_2}^2 - C_{r_1r_2}^2 - \sum C_{c_i r_j}^2 \\
 E_3^{(1*2,2)}(\rho_{c_1c_2r_2}) &= C_{c_1|c_2r_2}^2 - C_{c_1c_2}^2 - C_{c_1r_2}^2 \\
 E_3^{(1*2,2)}(\rho_{r_1c_2r_2}) &= C_{r_1|c_2r_2}^2 - C_{r_1c_2}^2 - C_{r_1r_2}^2, \quad (24)
 \end{aligned}$$

where the subscripts $i \neq j$ in the second equation. These multipartite entanglement indicators are nonnegative and invariant under local operations. The indicator $E_4^{(1,3,2)}$ can be used to characterize the genuine multipartite entanglement in the partition $c_1|r_1c_2r_2$, and $E_4^{(2,2,2)}$ can indicate the genuine block-block entanglement in the partition $c_1r_1|c_2r_2$ [42]. Moreover, the indicator $E_3^{(1,2,2)}$ is used to quantify the qubit-block entanglement in three-qubit mixed states [43–45].

In Fig.5, we plot the indicators of multipartite quantum correlations Q_4 and Q_3 as a function of the time evolution parameter κt in comparison to these multipartite entanglement indicators E_4 and E_3 in the output state $|\Phi_t\rangle$ with the initial state amplitude $\alpha = 1/\sqrt{10}$. As shown in the figure, the multipartite quantum correlation is correlated with the multipartite entanglement in every partition structure. However, the peak values of correlation and entanglement evolutions do not coincide completely. The reason is that quantum correlation and quantum entanglement are not equivalent in general. Particularly, in the dynamical procedure, the evolution of two-qubit entanglement can occur the phenomenon of entanglement sudden death [46–48], but the corresponding evolution of quantum correlation is always asymptotic.

V. DISCUSSION AND CONCLUSION

In a tripartite pure state $|\psi\rangle_{ABC}$, the operational interpretation of $D_{A|C}$ is that the discord is equal to the total consumption in the extended state merging (ESM) from A to B i.e., the resource needed in the preparation the quantum state ρ_{AB} plus the consumption in the state merging [10]. Our analysis and numerical results show that the quantum correlation distribution $D_{A|BC}^2 - D_{A|B}^2 - D_{A|C}^2$ is monogamous in three-qubit pure states, which can be explained that the square of the entanglement in the partition $A|BC$ is no less than the square sum of entanglement consumption in the ESM procedures from A to B and A to C .

Unlike quantum entanglement, quantum correlation characterized by the quantum discord is asymmetric, which means that $D_{A|B} \neq D_{B|A}$ in general. In the quantum correlation distribution $D_{A|BC}^2 - D_{A|B}^2 - D_{A|C}^2$, we have analyzed the SQDs with the measurement on different parties. However, when we consider the measurement on the same party, the correlation distribution $D_{B|C|A}^2 - D_{B|A}^2 - D_{C|A}^2$ may give a different value and its property is worth to further investigate.

In an N -qubit system, we can define different multipartite correlation indicators according to the structure of quantum correlation distribution. For the generalized N -qubit GHZ state $|GHZ_n\rangle = \alpha|0\rangle^{\otimes n} + \beta|1\rangle^{\otimes n}$, we have an effective indicator

$$Q_N^{[1*(N-1),2]} = D_{A_1|A_2\cdots A_N}^2 - \sum_i D_{A_1|A_i}^2 = S^2(A_1), \quad (25)$$

which detects the N -qubit quantum correlation. For the generalized N -qubit W state $|W_N\rangle = \alpha_1|10\dots 0\rangle + \cdots + \alpha_k|0\dots 1\dots 0\rangle + \cdots + \alpha_n|00\dots 1\rangle$, the indicator

$$\begin{aligned}
 Q_N^{[(N-2)*2,(N-1)]} &= D_{A_1\cdots A_{N-2}|A_{N-1}A_N}^2 - D_{A_1\cdots A_{N-2}|A_{N-1}}^2 \\
 &\quad - D_{A_1\cdots A_{N-2}|A_N}^2 \quad (26)
 \end{aligned}$$

always works because the subsystem $(A_1 \cdots A_{N-2})$ is equivalent to a logic qubit.

In conclusion, we have explored multipartite quantum correlations with the SQD. In tripartite quantum systems, a necessary and sufficient condition for the SQD being monogamous has been given. Particularly, in the case of three-qubit quantum states, we have given the measure and indicator for the genuine tripartite quantum correlation. Furthermore, we have investigated the correlation distribution of SQD in four-qubit pure states and constituted a set of correlation indicators. As an application, these indicators are used to characterize the evolution of multipartite cavity-reservoir systems. Our work may be helpful for further understanding of the properties of quantum correlation in multipartite systems.

Acknowledgments

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Appendix

A. Proof of the condition in Eq. (11)

We prove that the SQD is monogamous in three-qubit pure states if the condition in Eq. (11) is satisfied. The entanglement of formation is monotonic increasing function of the square of the concurrence. When the entanglement in the subsystem AC is not less than that in the subsystem AB , we can derive the following relations

$$\begin{aligned} E_f(C_{AC}^2) \geq E_f(C_{AB}^2) &\Rightarrow E_f(C_{AB|C}^2) \geq E_f(C_{AC|B}^2) \\ &\Rightarrow S(C) \geq S(B) \\ &\Rightarrow S(A|B) \geq 0, \end{aligned} \quad (27)$$

where we use the entanglement distributions $C_{AB|C}^2 = C_{AC}^2 + C_{BC}^2 + \tau_3$ and $C_{AC|B}^2 = C_{AB}^2 + C_{BC}^2 + \tau_3$ with τ_3 being the three-tangle [20]. According to these relations, we know that the sign of $E_f(AC) - E_f(AB)$ is the same as that of the conditional entropy $S(A|B)$. Therefore, the second term in the monogamous condition can be written as

$$\begin{aligned} &2S(A|B)[E_f(AC) - D(A|C)] \\ &= 2|S(A|B)|[|E_f(AC) - E_f(AB)| - |S(A|B)|]. \end{aligned} \quad (28)$$

When the condition in Eq. (11) holds, this term is directly proportional to the absolute value of the conditional entropy $S(A|B)$, which is obvious nonnegative. After considering the first term $M(E_f^2)$ being also nonnegative, we get that the SQD obeys the monogamous relation

$$D^2(A|BC) - D^2(A|B) - D^2(A|C) \geq 0. \quad (29)$$

This completes the proof.

B. Calculation of quantum correlation distributions

In Sec. IID, via a three-qubit mixed state $\rho_{ABC}(W)$, we show the quantum correlation distribution $D_{A|BC}^2 - D_{A|B}^2 - D_{A|C}^2$ can be negative in the mixed state case. Now, we give the calculation. Based on the Koashi-Winter relation in Eq. (3), we have the discord $D_{A|BC} = E_f(AE) - S(A|BC)$ in which the subsystem E is the environment degree of freedom which can purify the mixed state. Due to $\rho_{ABC}(W)$ is a rank-2 quantum state, the environment subsystem is equivalent to a logic qubit. We can get the $E_f(AE) = 0.06942$ by using the Wootters formula [19]. Then we have $D_{A|BC}^2 = 0.10845$. The discord for the qubits AB can be written in the form $D_{A|B} = E_f(A|CE) - S(A|B)$, in which the entanglement $E_f(A|CE)$ is available since the subsystem BE is equivalent

to a logic qubit and then we obtain $D_{A|B}^2 = 0.02368$. Similarly, we can derive $D_{A|C}^2 = 0.08994$. Substituting these SQDs into the correlation distribution, it is found that the value of the distribution is -0.00517 .

In Sec. III, we show the quantum correlation distribution $D_{A|BCD}^2 - D_{A|B}^2 - D_{A|C}^2 - D_{A|D}^2$ can be negative in the generalized four-qubit W state. The calculation is as follows. The multiqubit discord $D_{A|BCD}$ is equivalent to the von Neumann entropy $S(A)$ according to the property of pure state. By using the Koashi-Winter relation, we have the two qubit discord $D_{A|B} = E_f(A|CD) - S(A|B)$. Due to the subsystem CD being equivalent to a logic qubit, the $E_f(A|CD)$ is available, and then we can deduce further $D_{A|B}^2$. The case for other two-qubit discords are similar. When the parameters are chosen as $\vartheta_1 = 0.47\pi$ and $\vartheta_2 = \vartheta_3 = 0.28\pi$, the SQDs are $D_{A|BCD}^2 = 0.0053454$, $D_{A|B}^2 = 0.0018002$, $D_{A|C}^2 = 0.0018987$, and $D_{A|D}^2 = 0.0018592$, which results in the distribution $Q_4^{(1*3,2)} = -0.00021265$.

C. Calculation of the discord in cavity-reservoir systems

The density matrix of two-qubit X state can be written as

$$\rho_X^{AB} = \begin{pmatrix} a_{00} & 0 & 0 & a_{03} \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{12}^* & a_{22} & 0 \\ a_{03}^* & 0 & 0 & a_{33} \end{pmatrix}. \quad (30)$$

When the elements satisfy the following relations [41]:

$$|a_{12} + a_{03}| \geq |a_{12} - a_{03}| \quad (31a)$$

$$|\sqrt{a_{00}a_{33}} - \sqrt{a_{11}a_{22}}| \leq |a_{12}| + |a_{03}|, \quad (31b)$$

Chen *et al* proved that the optimal measurement for the quantum discord is σ_x .

In the output state $|\Phi_t\rangle$ of the cavity-reservoir systems, the reduced density of two cavity photons has the form

$$\rho_{c_1c_2} = \begin{pmatrix} \alpha^2 + \beta^2\chi^4 & 0 & 0 & \alpha\beta\xi^2 \\ 0 & \beta^2\xi^2\chi^2 & 0 & 0 \\ 0 & 0 & \beta^2\xi^2\chi^2 & 0 \\ \alpha\beta\xi^2 & 0 & 0 & \beta^2\xi^4 \end{pmatrix}. \quad (32)$$

It is obvious that the first relation is satisfied for the quantum state. Moreover, the second relation holds, due to

$$\begin{aligned} &|a_{12}| + |a_{03}| - |\sqrt{a_{00}a_{33}} - \sqrt{a_{11}a_{22}}| \\ &= \sqrt{(g_0 + g_1)^2} - \sqrt{g_0^2 + g_1^2} \geq 0, \end{aligned} \quad (33)$$

where $g_0 = \alpha\beta\xi^2$ and $g_1 = \beta^2\xi^2\chi^2$. Therefore, the optimal measurement for the quantum discord $D_{c_2|c_1}$ is $\sigma_x^{c_1}$, and the measurement operators are

$$M_0 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, M_1 = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad (34)$$

respectively. Then, according to the definition of the quantum discord in Eq. (2), we can get the value of $D_{c_2|c_1}^2$. For other

two-qubit quantum discords in the quantum correlation indicators Q_4 and Q_3 , we derive that the optimal measurement is

also σ_x . In a similar way, we can calculate these SQDs.

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